Solution of NLPP when constraints are not all Equality constraints.

Kuhn - Tucker Necessary Conditions for the optimality of the objective function in a GNLPP.

Let us consider the GNLPP

$$\max. Z = f(X) \qquad X = (x_1, x_2, \dots x_n) \in R^n$$

s.t.

$$g_i(X) \leq b_i \qquad i = 1, 2, \dots m$$

where, b_i's are constants.

Let $h_i(X) = g_i(X) - b_i \leq 0$

Introducing slack variables $S_1, S_2 \dots S_m$ (S_i ; $i = 1, 2, \dots m$), we have

$$h_i(X) + S_i^2 = 0$$

Thus, the given NLPP with inequalities constraints reduces to equality constraints as follows

$$\max. Z = f(X)$$

s.t.

$$h_i(X) + S_i^2 = 0$$

$$X \geq 0.$$

and hence it can be solved by using Lagrangian Multipliers $\lambda_1, \lambda_2 \dots \lambda_m$

Let us define the Lagrangian function

$$L(X, S, \lambda) = f(X) - \sum_{i=1}^{m} \lambda_i \{ h_i X + s_i^2 \}$$

where,

$X = (x_1, x_2, \dots x_n) \in R^n$

$\lambda = (\lambda_1, \lambda_2 \dots \lambda_m) \in R^m$

$S = (S_1, S_2 \dots S_m) \in R^m$

Let L, f and h_i are all differentiable partially w.r.t. $x_1, x_2, \dots x_n, \lambda_1, \lambda_2 \dots \lambda_m$; $S_1, S_2 \dots S_m$

The necessary conditions for the stationary points are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x_j} = 0 \quad \text{--- (1)}$$

$\forall j = 1, 2, \dots n$

$$\frac{\partial L}{\partial \lambda_i} = -[h_i(X) + s_i^2] = 0 \; ; \; i = 1, 2, \dots m \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial s_i} = -2 s_i \lambda_i = 0 \; ; \; i = 1, 2, \dots m \quad \text{--- (3)}$$

From ③, we have

either $s_i = 0$ or $\lambda_i = 0$

If $s_i = 0$

Then ② $\Rightarrow$ $h_i(x) = 0$

$\therefore \lambda_i = 0$ or $s_i = 0$

$\Rightarrow \lambda_i = 0$ or $h_i(x) = 0$ ——— ④

$\therefore \lambda_i h_i(x) = 0$

$\because s_i^2 \geq 0$

② $\Rightarrow h_i(x) \leq 0$

when $h_i(x) < 0$

④ $\Rightarrow \lambda_i = 0$ and when $\lambda_i > 0$, $h_i(x) = 0$

$\therefore$ ④ $\Rightarrow h_i(x) = 0 \Rightarrow \lambda_i$ is unrestricted in sign.

If $\lambda_i \neq 0$ then $s_i = 0$

② $\Rightarrow h_i(x) = g_i(x) - b_i = 0$

i.e. $g_i(x) = b_i$

Hence Kuhn-Tucker necessary conditions for the point x to be a point of maximum for $f(X)$, subject to $h_i(X) = g_i(X) - b_i \leq 0$ are

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x_j} = 0 \quad \forall\, j = 1, 2, 3 \ldots n \\ &\lambda_i h_i(X) = 0 \\ &h_i(X) \leq 0 \quad \text{and} \quad \lambda_i \geq 0 \quad i = 1, 2, \ldots m \end{aligned}}$$

Illy, the Kuhn-Tucker necessary condition for the point x to be a minimum of $f(X)$, subject to $h_i(X) = g_i(X) - b_i \geq 0$ are

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x_j} = 0 \quad \forall\, j = 1, 2, \ldots n \\ &\lambda_i h_i(X) = 0 \\ &h_i(X) \geq 0 \quad \text{and} \quad \lambda_i \geq 0 \quad i = 1, 2, \ldots m \end{aligned}}$$

Kuhn-Tucker sufficient condⁿ for the optimality of the objective function of a GNLPP with inequality condⁿˢ.

(1) The Kuhn-Tucker necessary condⁿˢ for NLPP, max. $Z = f(X)$; $X \in R^n$

s.t.

$$h_i(X) = g_i(X) - b_i \leq 0 \quad i = 1, 2, \dots m$$

$$X \geq 0$$

are also the sufficient condⁿˢ for maximum of $f(X)$ if

i) $f(X)$ is concave

2) $h_i(X)$ (i.e. $g_i(x)$) are convex funⁿ of X

i.e. $-h_i(X)$ are also concave funⁿ of X

(2) The Kuhn-Tucker N/C for NLPP,

min $f(X)$.

s.t.

$$h_i(X) = g_i(X) - b_i \geq 0 \quad i = 1, 2, \dots m$$

$$X \geq 0.$$

are also the sufficient condⁿ for minimum of $f(X)$ if

i) $f(X)$ is convex

2) $h_i(X)$ are ~~also~~ ~~convex~~ Concave fun²s of x.

ie. $-h_i(X)$ are also convex fun² of X.

(For more details Refer page 839, 840) of R.K. Gupta).